### **Efficient Statistics With Unknown Truncation,**

Polynomial Time Algorithms, Beyond Gaussians

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#### Anay Mehrotra



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# ₩ FOCS 2024

#### **Estimation from Truncated Data**



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 $O(d^2/\varepsilon^2)$  algorithms with "known" *S*\*

- Gaussians Daskalakis, Gouleakis, Tzamos, and Zampetakis (FOCS 2018)
- Certain exponential families Lee, Wibisono, Zampetakis (NeurIPS 2023)

Gaussian Surface Area  $\Gamma(S)$ : S

Surface area with respect to the Gaussian measure





 $d^{\operatorname{poly}(\Gamma(S^*)/\varepsilon)}$  algorithms with unknown  $S^*$ 

• *Diagonal* gaussians

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- **Q1:** Estimating *general* Gaussians with unknown *S*\*?
- **Q2:** poly( $d/\varepsilon$ ) time estimation for halfspaces?

## **Results for Gaussians**

**Informal Theorem 1:** In  $d^{\text{poly}(\Gamma(S^*)/\varepsilon)}$  time, we can find  $(\mu, \Sigma)$ , s.t., with 99% probability

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Both results extend to truncated linear regression with Gaussian covariates with unknown S\*







Degree-*p* polynomials poly( $\varepsilon$ )-approximate *S*<sup>\*</sup> with respect to  $\mathcal{E}(\theta^*)$  in *L*<sub>2</sub>-norm





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for all sets  $T \subseteq \mathbb{R}^d$ ,  $e^{-C} \cdot \mathcal{E}(\theta_0; T) \leq \mathcal{E}(\theta^*; T) \leq e^C \cdot \mathcal{E}(\theta_0; T)^{1/C}$ 



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- *n* iid positive samples and
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Distribution of positive samples "Approximate" unlabeled distribution Log-concave bridge distribution  $\mathcal{B}$ Mixture constructed

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Axis-Aligned Rectangle

**Folklore Method** to learning from only positive samples





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## Summary and Open Problems

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## Thank You!